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On projective space bundles with nef normalized tautological divisor

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Abstract
In this poster, I explain about the structure of projective bundles with nef normalized tautological divisor.

Definition of normalized tautological divisor

X : *sm. proj. var.* / $k = \bar{k}$, $\mathcal{E}$: *vect. bdle of rank* r,

$\pi : \mathbb{P}_X(\mathcal{E}) \to X$: *projectivization*, $\xi_{\mathcal{E}}$: *tautol. div.*

$(\Diamond)$ $\Lambda_{\mathcal{E}} := \xi_{\mathcal{E}} - \frac{1}{r}\pi^*(\mathbf{det}(\mathcal{E}))$: *norm. taut. div.*

$r\Lambda_{\mathcal{E}} = r\xi_{\mathcal{E}} - \pi^*(det(\mathcal{E})) = -K_\pi$: *rel.anti-can.div.*

Problem

Study the structure of projective space bundles with nef normalized tautological divisor.

Ample case, Kollár-Miyaoka-Mori [KMM]

$\pi : Y \to X$: *gen.smooth mor.* $\Rightarrow -K_\pi$: *not ample*
In particular, $\Lambda_{\mathcal{E}}$ cannot be ample.

Nef and big case, Theorem1

If $\Lambda_{\mathcal{E}}$ is nef $\Rightarrow$ $\Lambda_{\mathcal{E}}$ cannot be big.

Trivial example ($\Lambda_{\mathcal{E}}$ is semiample)

If $\mathcal{E} \cong \mathcal{O}_X^r$, then $r\Lambda_{\mathcal{E}} = p^*(-K_{\mathbb{P}^{r-1}})$ is basepoint-free where $p : \mathbb{P}_X(\mathcal{E}) \cong X \times \mathbb{P}^{r-1} \to \mathbb{P}^{r-1}$ is the second projection.

Semiample case, Theorem2

1. $char(k) = 0$ and $\Lambda_{\mathcal{E}}$ is semiample
   $\Rightarrow \exists f : X' \to X$; finite étale morphism
   s.t. $f^*\mathcal{E}$ is trivial up to twist by some line bundle.
2. $char(k) > 0$ and $\Lambda_{\mathcal{E}}$ is semiample
   $\Rightarrow \exists f : X' \to X$; finite surj. mor. from normal var.
   s.t. $f^*\mathcal{E}$ is trivial up to twist by some line bundle.

Nef case 1, Miyaoka [M], Nakayama [N]

X : *d-dim. sm. proj. var.* / $\mathbb{C}$, $\mathcal{E}$: *vect. bdle of rank* r,
Then the following conditions are equivalent:

1. $\Lambda_{\mathcal{E}}$ is nef;
2. $\mathcal{E}$ is A-semistable and
$$(c_2(\mathcal{E}) - \frac{2r}{r-1}c_1^2(\mathcal{E})).A^{d-2} = 0$$
   for an ample divisor A;
3. $$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_l = \mathcal{E} \ : \ \textit{filt.of subbdles}.$$
   s.t. $\mathcal{E}_i/\mathcal{E}_{i-1}$ are induced from a repre. $\pi_1(X) \to PU(r)$ and $\mu(\mathcal{E}_i/\mathcal{E}_{i-1})$ are num. equiv. to $\mu(\mathcal{E}) := c_1(\mathcal{E})/rank(\mathcal{E})$ for any i.

Corollary. *X, $\mathcal{E}$ as above.*

1. *Assume that X is simply connected and $\Lambda_{\mathcal{E}}$ is nef. Then $\mathcal{E}$ is trivial up to twist by a line bundle.*
2. *Assume that $\mathbb{P}_X(\mathcal{E})$ is log Fano and $\Lambda_{\mathcal{E}}$ is nef. Then X is log Fano and $\mathbb{P}_X(\mathcal{E}) \cong X \times \mathbb{P}^{n-1}$.*

Nef case 2, Theorem3

X : *d-dim. sm. proj. var.of pos.char.*
$\mathcal{E}$: *vect. bdle of rank* r,
Then the following conditions are equivalent:

1. $\Lambda_{\mathcal{E}}$ is nef;
2. $\mathcal{E}$ is strongly A-semistable and
$$(c_2(\mathcal{E}) - \frac{2r}{r-1}c_1^2(\mathcal{E})).A^{d-2} = 0$$
   for an ample divisor A;

Remark. This is proved by Y. Miyaoka[M] in the curve case and A. Langer[L] in the case where $det(\mathcal{E})$ is trivial.

Corollary. *Let S be a K3 surface or an Enriques surface then Ω_S is not nef.*

Tangent bundle, Theorem4

1. $\Lambda_{\mathcal{T}_X}$ is nef $\Rightarrow$ X contains no rational curve;
2. S : sm. proj. surf. / $\mathbb{C}$. If $\Lambda_{\mathcal{T}_S}$ nef
   $\Rightarrow \exists f : A \to S$: étale covering from abelian surface A.

Remark. This is proved by I. Biswas[B] independently and recently proved by P. Jahnke and I. Radloff in arbitrary dimension.

References

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